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Fixed Points for Directional Multi-Valued $k(\cdot)$-Contractions

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Abstract. A fixed point theorem for directional multi-valued $k(\cdot)$-contractions acting in a complete metric space is established which extends similar results both for $k(\cdot)$-contractions and directional contractions. Such theorem enables to obtain fixed points theorems for the former class of set-valued maps from those valid for the latter one without metrical convexity or proximinality assumptions, thereby contributing to unify the current setting of the theory. Connections with several recent advances on this subject are also examined.

Mathematics Subject Classifications (2000): 47H10; 54H25

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1. Introduction

Fixed point theory, as a relevant topic both in pure and applied mathematics, is a flourishing branch of nonlinear analysis with many directions of development.

The present work is focused on fixed point theory for contractive maps in the metric space setting. Within such context, the development of the theory in the last decades witnessed, among many others, two fundamental advances that in turn stimulated a good deal of recent investigations on the subject. The first one is the extension of the celebrated Banach contraction principle to multi-valued maps, known as Nadler fixed point theorem (see [10, 15, 17]), that put to the center of this kind of researches set-valued maps, a tool which, despite its utility, was not yet adequately considered. Subsequently, in [18] some fixed point results were achieved which took under examination multi-valued maps with variable contraction factors, thereby extending analogous results for single-valued maps (see [6]). This second advance phase culminated with the Reich's conjecture (still now not completely answered) asking whether, in force of a rather general lim sup condition on the contraction factor (as a function of distance), the compactness hypothesis on the values taken by a multi-valued contraction could be relaxed (see [19]). Recent attempts to drop out such compactness hypothesis led to modify the lim sup condition, by making it stronger (see [7, 11, 12, 16]).

Another direction of research (see [20]) led to extend to the set-valued map setting a previous fixed point result valid for single-valued maps (see [8, 9, 14]) with only directional contraction properties. Recently, in [22] such investigations were refined by some new achievement and it was shown that several already existent results, even relating to classic (not directional) contractive maps, could be improved via this approach.

The present paper intends to carry on such development of the theory by examinating directional multi-valued contractions with variable contraction factor. Thus the two mentioned lines of current investigations are, to some extent, unified. In this spirit, a fixed point theorem is presented which extends both the ones in [20, 22] and other theorems obtained in [11, 16]. Other connections with related results are also examinated.

A remarkable feature of the issue is that, as it happens for many existence questions in nonlinear analysis, fixed points are obtained as a global minimum point for proper perturbations of the displacement function associated to a map, emphasizing the variational aspect of the problem. This fact must be not surprising in as much as it is well known that fixed point theory and optimization are connected by close relationships. This intriguing link appeared in all its profundity when the Ekeland variational principle (here the main tool of analysis, along with some its variant) and the Caristi fixed point theorem were proved both to be equivalent formulations of completeness for a metric space (see [5]). In the same vein, more recently the mentioned variational principle has been reformulated as an existence result for global minima in the Takahashi theorem (see [23]).

2. Directional Multi-valued $k(\cdot)$-Contractions

Throughout the paper $(\mathbb{E}, d)$ denotes a metric space, whereas the class of all its nonempty closed and bounded subsets is denoted by $CB(\mathbb{E})$. $\mathbb{R}_+$ is the set of all nonnegative real numbers. In order to introduce multi-valued contractions one needs the Hausdorff metric $\mathcal{H} : CB(\mathbb{E}) \times CB(\mathbb{E}) \longrightarrow \mathbb{R}_+$ induced by d over $CB(\mathbb{E})$. namely, given a pair A, B of elements of $CB(\mathbb{E})$, $\mathcal{H}(A, B) = \max\{H_+(A, B), H_+(B, A)\}$, where

$$H_+(A, B) = \sup_{b \in B} d(b, A) = \sup_{b \in B} \inf_{a \in A} d(b, a).$$

Nonetheless, to present results in their full generality, instead of $\mathcal{H}$ merely H_+ will be mainly used in the sequel. Given any pair x and y of distinct points of $\mathbb{E}$, by $]x, y[$ the set will be denoted consisting of all the points $z \in \mathbb{E} \setminus \{x, y\}$ such that $d(x, z) + d(z, y) = d(x, y)$.

REMARK 2.1. It is not difficult to show that, given any pair $A, B \in CB(\mathbb{E})$ and any $x \in \mathbb{E}$, it holds $d(x, B) \leqslant d(x, B) + H_+(A, B)$.

Following an already proposed scheme of analysis (see, for instance, [8, 9, 13]), existence of fixed points will be investigated by means of a variational approach. More precisely, they will be obtained as a global minimum point for certain perturbations of the so called "displacement function" associated to a multi-valued contraction. To this end, the following lemma, whose proof can be easily drawn by a perusal of the proof of Theorem 2.1 in [24], will be of use.

LEMMA 2.1. *Let K be a closed subset of* $\mathbb{E}$. *If a multi-valued map* $F : K \longrightarrow CB(\mathbb{E})$ *is upper semicontinuous (for short, u.s.c.), then its displacement function* $f : K \longrightarrow \mathbb{R}_+$ *defined by* $f(x) = d(x, F(x))$, *is lower semicontinuous (for short, l.s.c.).*

The notion of upper semicontinuity is, of course, the usual one for set-valued maps (as one reads e.g. in [2]).

The next definition introduces to the notion which plays the major role in this section.

DEFINITION 2.1. Let $K \subseteq \mathbb{E}$. A map $F : K \to CB(\mathbb{E})$ is called a *directional multi-valued* $k(\cdot)$*-contraction* if there exists $\alpha \in (0, 1], \mathrm{a} : (0, +\infty) \to [\alpha, 1]$ and $k : (0, +\infty) \to [0, 1)$ such that for every $x \in K$, with $x \notin F(x)$, there is $y \in K \setminus \{x\}$ satisfying the inequalities

$$a(d(x, y))d(x, y) + d(y, F(x)) \leqslant d(x, F(x))$$

and

$$H_+(F(y), F(x)) \leqslant k(d(x, y))d(x, y).$$

It is clear that such class includes in particular maps F for which the second inequality in Definition 2.1 is satisfied with $H_+(F(y), F(x))$ replaced by $\mathcal{H}(F(y), F(x))$.

EXAMPLE 2.1. Let $\mathbb{E} = [0, 1] \times [0, 1]$ and let $\mathbb{Q}$ denote the set of all rational numbers. Consider the metric space $(\mathbb{E}, d)$, where d denotes the Euclidean distance, and the map $F : \mathbb{E} \to CB(\mathbb{E})$ (actually single-valued) defined by

$$F(x_l, x_2) = \begin{cases} (1, 1), & \text{if } (x_1, x_2) \in \mathbb{E} \cap (\mathbb{Q} \times \mathbb{Q}), \\ (0, 0), & \text{otherwise.} \end{cases}$$

The map F satisfies Definition 2.1 with $\alpha = 1$, $a \equiv 1$ and $k \equiv 0$. Indeed, without difficulty one sees that for any point $(x_1, x_2) \in \mathbb{E} \cap (\mathbb{Q} \times \mathbb{Q}) \setminus \{(1, 1)\}$ the line segment connecting (x_1, x_2) with $(1, 1) = F(x_1, x_2)$ contains a point $(y_1, y_2) \in \mathbb{E} \cap (\mathbb{Q} \times \mathbb{Q}) \setminus \{(x_1, x_2), (1, 1)\}$, so that it holds

$$d((x_1, x_2), (y_1, y_2)) + d((y_1, y_2), (1, 1)) = d((x_1, x_2), F(x_1, x_2))$$

and $d(F(y_1,y_2),F(x_1,x_2))=0$. Analogously, for any point $(x_1,x_2)\in \mathbb{E}\setminus(\mathbb{Q}\times\mathbb{Q})$, the line segment connecting (x_1,x_2) with $(0,0)=F(x_1,x_2)$ contains a point $(y_1,y_2)\in\mathbb{E}\setminus((\mathbb{Q}\times\mathbb{Q})\cup\{(x_1,x_2)\})$, so that it holds

$$d((x_1,x_2),(y_1,y_2))+d((y_1,y_2),(0,0))=d((x_1,x_2),F(x_1,x_2))$$

and $d(F(y_1,y_2),F(x_1,x_2))=0$. Nonetheless, F fails clearly to be a metric contraction of $\mathbb{E}$.

The main theorem of the paper is stated below.

THEOREM 2.1. *Let K be a closed nonempty subset of a complete metric space $(\mathbb{E},d)$ and let $F:K\to CB(\mathbb{E})$ be an u.s.c. directional multi-valued $k(\cdot)$-contraction. Assume that there exist $x_0\in K$ and $\delta>0$ such that $d(x_0,F(x_0))\leqslant\alpha\delta$ and*

$$\sup_{t\in(0,\delta]}k(t)<\inf_{t\in(0,\delta]}a(t),$$

where $\alpha\in(0,1]$, a and k are the constant and the functions occuring in the definition of directional multi-valued $k(\cdot)$-contraction. Then F admits a fixed point.

Proof. By hypothesis, there exists $\beta>0$ and $\delta>0$ such that

$$\sup_{t\in(0,\delta]}[k(t)-a(t)]\leqslant\sup_{t\in(0,\delta]}k(t)-\inf_{t\in(0,\delta]}a(t)\leqslant-\beta. \tag{1}$$

Since F is u.s.c., the displacement function $f:K\to\mathbb{R}_+$ associated to F is l.s.c. in K, according to Lemma 2.1. Since K is complete if equipped with the metric induced by d, and $f(x_0)\leqslant\alpha\delta$, then it is possible to apply the Ekeland variational principle (see, for instance, [13]) around x_0, to get for any $\lambda>0$ the existence of $x_\lambda\in K$ such that $f(x_\lambda)\leqslant f(x_0)$ and

$$f(x_\lambda)<f(x)+\frac{\alpha\delta}{\lambda}d(x_\lambda,x),\quad\forall x\in K\setminus\{x_\lambda\}. \tag{2}$$

Suppose ab absurdo that $f(x_\lambda)>0$ for every $\lambda>0$. Take $\lambda=\frac{2\alpha\delta}{\beta}$. Since F is a directional multi-valued $k(\cdot)$-contraction, according to Definition 2.1 there exists $y\in K\setminus\{x_\lambda\}$ satisfying the inequalities

$$a(d(x_\lambda,y))d(x_\lambda,y)+d(y,F(x_\lambda))\leqslant f(x_\lambda), \tag{3}$$

and

$$H_+(F(y),F(x_\lambda))\leqslant k(d(x_\lambda,y))d(x_\lambda,y).$$

From inequality (3), being $\alpha\leqslant a(d(x_\lambda,y))$, it follows

$$d(x_\lambda,y)\leqslant\alpha^{-1}f(x_\lambda)\leqslant\alpha^{-1}f(x_0),$$

so $0<d(x_\lambda,y)\leqslant\delta$, Moreover, one has

$$d(y,F(x_\lambda))\leqslant f(x_\lambda)-a(d(x_\lambda,y))d(x_\lambda,y).$$

In the light of Remark 2.1, from the last inequality one obtains

$$\begin{aligned}f(y)&\leqslant d(y,F(x_\lambda))+H_+(F(y),F(x_\lambda))\\&\leqslant f(x_\lambda)+[k(d(x_\lambda,y))-a(d(x_\lambda,y))]d(x_\lambda,y).\end{aligned}$$

Putting $x = y$ in inequality (2) and taking into account inequality (1) along with $0 < d(x_\lambda, y) \leqslant \delta$, one finds

$$\begin{aligned} f(x_\lambda) &< f(y) + \frac{\alpha\delta}{\frac{2\alpha\delta}{\beta}} d(x_\lambda, y) \\ &\leqslant f(x_\lambda) + \left\{ [k(d(x_\lambda, y)) - a(d(x_\lambda, y))] + \frac{\beta}{2} \right\} d(x_\lambda, y) \\ &\leqslant f(x_\lambda) - \frac{\beta}{2} d(x_\lambda, y) < f(x_\lambda), \end{aligned}$$

which yields an absurdum. Therefore it must be $f(x_\lambda) = 0$ and this completes the proof. □

REMARK 2.2. Notice that the contractive condition $k(\cdot) < 1$, according to Definition 2.1, plays an essential role in Theorem 2.1, so the latter one cannot be extended to merely directional multi-valued Lipschitz maps. Indeed, the first inequality to be satisfied in Definition 2.1, along with the triangle inequality, forces $a(d(x, y)) \leqslant 1$. Therefore, the condition $\sup_{t \in (0,\delta]} k(t) < \inf_{t \in (0,\delta]} a(t)$ entails $k(t) < 1$ for every $t \in (0, \delta]$.

In order to draw a straightforward consequence of Theorem 2.1, that will serve to put it in comparison with other similar results, let us recall that a map $F: K \to CB(\mathbb{E})$ is said to have the almost fixed point property in K provided that $\inf_{x \in K} d(x, F(x)) = 0$.

COROLLARY 2.1. *Let $(\mathbb{E}, d)$ be a complete metric space and let $K \subseteq \mathbb{E}$ be closed and nonempty. If an u.s.c. directional multi-valued $k(\cdot)$-contraction satisfies the condition*

$$(C) \quad \limsup_{s \to 0^+} k(s) < \liminf_{s \to 0^+} a(s),$$

and it has the almost fixed point property in K, then it admits a fixed point (in K).

Proof. Observe that condition (C) implies the existence of $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\sup_{(0,\delta_1]} k(s) < \inf_{(0,\delta_2]} a(s),$$

so that it suffices to take $\delta = \min\{\delta_1, \delta_2\}$ to get satisfied the related in equality in the hypothesis of Theorem 2.1. Besides, since $\inf_{x \in K} f(x) = 0$, there exists a proper $x_0 \in K$ such that $f(x_0) \leqslant \alpha\delta$. □

Theorem 2.1 extends the fixed point theorem that was recently established in [22] for directional multi-valued contractions with constant contraction factor k and a. Indeed, in such event, it results $a \equiv \alpha \leqslant 1$ and $k < \alpha$

independently on t, so that it is always possible, taken an arbitrary $x_0 \in K$, to find a proper $\delta > 0$ such that $d(x_0, F(x_0)) \leqslant \alpha\delta$.

REMARK 2.3. Whereas for multi-valued $k(\cdot)$-contractions (see Section 3) the hypothesis of Theorem 2.1 about upper semicontinuity of F would be redundant whenever H_+ could be replaced by $\mathcal{H}$ (in such event f turns out to be Lipschitz continuous), for directional multi-valued $k(\cdot)$-contractions such hypothesis seems essential even when Definition 2.1 is satisfied with H_+ replaced by $\mathcal{H}$. However, according to the proof of Theorem 2.1 the role of upper semicontinuity consists only in ensuring f to be l.s.c. and therefore such hypothesis can be replaced with any other forcing the same property on the displacement function.[1]

Within the proposed approach it is possible also to derive a fixed point result for multi-valued contractions having a variable contractive behavior, which is locally described by a condition expressed in terms of a properly defined directional derivative.

Let a closed subset $K \subseteq \mathbb{E}$, a set-valued map $F: K \to CB(\mathbb{E})$, a pair of elements $x \in K$ and $z \in \mathbb{E}$ be given. Then the value

$$DF^{\downarrow}(x;z) = \begin{cases} 0, & \text{if } z = x, \\ +\infty, & \text{if }]x,z[\cap K = \emptyset, \\ \inf_{y \in]x,z[\cap K} \frac{H_+(F(y),F(x))}{d(y,x)} & \text{if }]x,z[\cap K \neq \emptyset \end{cases}$$

will be called *weak directional derivative* of F at x in the direction z. This kind of derivative (with $\mathcal{H}$ instead of H_+), firstly employed in [20] in connection with fixed point theory, minorizes the directional derivative $\underline{D}F(x;z)$ proposed in [8] for the single-valued case, so that it allows to achieve sharper results.

In view of the next statement, let us recall that a set $A \subseteq \mathbb{E}$ is called proximinal provided that for every $x \in \mathbb{E}$ there exists $a \in A$ such that $d(x,a) = d(x,A)$. The set consisting of all such $a \in A$ will be indicated henceforth by $\mathcal{P}(x,A)$. For instance, any nonempty closed convex subset of a reflexive Banach space is proximinal.

THEOREM 2.2. *Let* $(\mathbb{E}, d)$ *be a complete metric space, let K be a closed subset of* $\mathbb{E}$ *and let* $F: K \to CB(\mathbb{E})$ *be an u.s.c. map taking proximinal values. If there exists* $\sigma \in (0,1)$ *such that for each* $x \in K$, *with* $x \notin F(x)$,

$$\inf_{z \in \mathcal{P}(x,F(x))} DF^{\downarrow}(x;z) \leqslant \sigma,$$

then F admits a fixed point.

[1] In several formulations of fixed point theorems f is directly assumed to be l.s.c. (see, for instance, [20]).

Proof. It suffices to note that under the above assumptions map F is a directional multi-valued $k(\cdot)$-contraction in the sense of Definition 2.1. Indeed, for each $x \in K$, with $x \notin F(x)$, there is $z \in \mathcal{P}(x, F(x))$ such that $DF^{\downarrow}(x; z) \leqslant \frac{1}{2}(\sigma + 1)$ and, consequently, there is $y \in]x, z[\cap K$ such that $d(x, y) + d(y, F(x)) \leqslant d(x, y) + d(y, z) = d(x, F(x))$ and $H_+(F(y), F(x)) \leqslant \frac{\sigma+3}{4} d(x, y)$. So, F fulfills Definition 2.1 with $a \equiv \alpha = 1$ and $k = \frac{\sigma+3}{4}$. Besides, since k is a constant less than 1, then taken an arbitrary $x_0 \in K$ there is no difficulty in finding $\delta > 0$ in such a way that all the hypotheses of Theorem 2.1 are fulfilled. □

It may be worth noticing that from Theorem 2.2 one can derive also the set-valued extension presented in [20] of an earlier fixed point theorem for directional contractions in the Banach space setting due to Kirk and Ray (see [14]).

3. Some Related Results

Let us recall that a map $F: \mathbb{E} \to CB(\mathbb{E})$ is said to be a multi-valued $k(\cdot)$-contraction if there exists $k: (0, +\infty) \to (0, 1)$ such that for every $x \notin F(x)$ and for every $y \in F(x)$ it holds

$$H_+(F(y), F(x)) \leqslant k(d(x, y)) d(x, y).$$

Under a rather general condition on contraction factor $k(\cdot)$ the next proposition, which is a slight refinement of Lemma 1 in [21] and is fully proved here for the sake of completeness, shows that multi-valued $k(\cdot)$-contractions have the almost fixed point property.

PROPOSITION 3.1. *If a multi-valued $k(\cdot)$-contraction F has a factor $k(\cdot)$ satisfying the condition*

$$(R) \ \limsup_{s \to t^+} k(s) < 1, \quad \forall t \in (0, +\infty),$$

then F has the almost fixed point property.

Proof. Take arbitrarily $x_0 \in E$ and $x_1 \in F(x_0)$. If $x_1 = x_0$, the thesis is trivially verified. Otherwise, it is possible to contruct inductively a sequence $(x_n)_{n \in \mathbb{N}}$ *in* $\mathbb{E}$ such that $x_{n+1} \in F(x_n)$ and $d_n = d(x_{n+1}, x_n)$ is decreasing. Indeed, setting $d_0 = d(x_0, x_1)$, since it is

$$\sup_{x \in F(x_0)} d(x, F(x_1)) \leqslant k(d_0) d_0 < \frac{1}{2}(1 + k(d_0)) d_0 < d_0,$$

then $d(x_1, F(x_1)) < \frac{1}{2}(1 + k(d_0)) d_0$ and hence it is possible to find $x_2 \in F(x_1)$ such that $d(x_2, x_1) < \frac{1}{2}(1 + k(d_0)) d_0$. By proceeding in this vein, set $d_1 = d(x_2, x_1)$, since it is

$$\sup_{x \in F(x_1)} d(x, F(x_2)) \leqslant k(d_1) d_1 < \frac{1}{2}(1 + k(d_1)) d_1 < d_1,$$

then $d(x_2, F(x_2)) \leqslant \frac{1}{2}(1 + k(d_1))d_1$ so one gets the existence of $x_3 \in F(x_2)$ such that $d(x_3, x_2) < \frac{1}{2}(1 + k(d_1))d_1$. Observe that, letting $d_2 = d(x_3, x_2)$, one has

$$d_2 < \frac{1}{2}(1 + k(d_1))d_1 < d_1 < \frac{1}{2}(1 + k(d_0))d_0 < d_0.$$

Thus, by induction, one obtains a strictly decreasing sequence $(d_n)_{n\in\mathbb{N}}$, with $d_n = d(x_{n+1}, x_n)$ and $x_{n+1} \in F(x_n)$. To prove the thesis one needs to show that $l = \lim_{n\to\infty} d_n$ (which exists by monotonicity) reduces to 0. Assume to the contrary that $l > 0$. From the inequality

$$2d_{n+1} < (1 + k(d_n))d_n,$$

which holds by induction for every $n \in \mathbb{N}$, one obtains passing to the limsup as $n \to \infty$

$$2l \leqslant (1 + \limsup_{s\to l^+} k(s))l < 2l,$$

which leads to an absurdum. Thus it must be $l = 0$. As $d(x_n, F(x_n)) \leqslant d(x_n, x_n + 1)$, it follows

$$\inf_{x\in\mathbb{E}} d(x, F(x)) = \lim_{n\to\infty} d(x_n, F(x_n)) = 0,$$

which completes the proof. □

Notice that in the above result the metric space $(\mathbb{E}, d)$ needs not be complete. Besides, F needs not be u.s.c., in contrast to the statement of Lemma 1 in [21], where the use of the Hausdorff metric leads to deal with u.s.c. maps only. Below Proposition 3.1 is employed to derive, within the proposed approach, a fixed point theorem which was established in order to answeiuthe Reich's conjecture.[2]

THEOREM 3.1. (Mizoguchi–Takahashi) *Let* $(\mathbb{E}, d)$ *be a complete metric space and let* $F : \mathbb{E} \to CB(\mathbb{E})$ *be an u.s.c. multi-valued* $k(\cdot)$*-contraction. If* $k(\cdot)$ *is a function such that* $\limsup_{s\to t^+} k(s) < 1$ *for every* $t \in [0, +\infty)$, *then* F *admits a fixed point.*

Proof. Since F is a $k(\cdot)$-contraction whose contraction factor satisfies, in particular, condition (R), by virtue of Proposition 3.1 F has the almost fixed point property. Moreover, by hypothesis, there exists $\alpha > 0$ such that $\limsup_{s\to 0^+} k(s) < \alpha < 1$. Given $x \notin F(x)$, it is thereby possible to find $y \in F(x)$ such that

$$d(x, y) \leqslant \frac{d(x, F(x))}{\alpha},$$

[2]As remarked clearly in [12, 21], the Reich's conjecture still remains unanswered in its original formulation, because the contraction factor $k(\cdot)$ should be supposed to satisfy only condition (R), unlike in the statement of Theorem 3.1.

so $\alpha d(x, y) + d(y, F(x)) \leqslant d(x, F(x))$. It remains to apply Corollary 2.1 with $K = \mathbb{E}$ and $a \equiv \alpha$. □

As one observes at once, Theorem 3.1 encompasses Theorem 1.2 of [11] in which instead of the lim sup condition one finds the increasing monotonicity of $k(\cdot)$ over $(0, +\infty)$ as an hypothesis.

It is worth observing that the above stated version of the Mizoguchi-Takahashi fixed point theorem, unlike the original one (see [11, 16]), does not require a contractive behavior of F also for pairs $x, y \in \mathbb{E}$ such that $y \notin F(x)$.

REMARK 3.1. Notice that Theorem 3.1 has been obtained by Corollary 2.1 without invoking any metrical convexity assumption on the space $(\mathbb{E}, d)$. Let us recall that a metric space $(\mathbb{E}, d)$ is called metrically convex, according to [4], provided that for any pair x, y of distinct elements of $\mathbb{E}$, the subset $]x, y[$ of $\mathbb{E}$ is nonempty. A special class of metrically convex metric spaces is that of uniformly Lipschitz-connected metric spaces of rank 1, which was involved in fixed point theory already in [5]. The latter class includes, in particular, all normed spaces.

It is clear that any multi-valued $k(\cdot)$-contraction F fulfills automatically Definition 2.1 whenever $(\mathbb{E}, d)$ is metrically convex and F takes proximinal values. Therefore, in such setting. Theorem 3.1 becomes a straightforward consequence of Corollary 2.1. In a similar manner, Banach and Nadler fixed point theorems have been distilled in the same setting from corresponding fixed point results valid for directional (single and set-valued, respectively) contractions. Nevertheless Theorem 2.1 shows that this can be done without invoking metrical convexity and proximinality assumptions, emphasizing the unifying role of directional contractions.

In the next example a map is exhibited which satisfies Definition 2.1 but is defined in a non metrically convex metric space.

EXAMPLE 3.1. Let L_- and L_+ denote the two closed line segments in the Euclidean plane $\mathbb{R}^2$ connecting (−1, 0) with (0, 1), and (0, 1) with ($\frac{1}{2}$, 0), respectively. Letting $\mathbb{E} = L_- \cup L_+$, consider the metric space $(\mathbb{E}, d)$, where d denotes the Euclidean distance, and the map $F\colon \mathbb{E} \to CB(\mathbb{E})$ (actually single-valued) defined by

$$F(x_1, x_2) = \begin{cases} (x_1, x_2), & \text{if } x \geqslant 0, \\ \left(-\frac{x_1}{2}, x_2\right), & \text{otherwise}. \end{cases}$$

F is not a metric contraction of $\mathbb{E}$ because all points in L_+ are fixed points for F. Let us show that F satisfies Definition 2.1 with $\alpha = \frac{3-\sqrt{5}}{2\sqrt{2}}$, $a \equiv \alpha$ and $k \equiv \frac{\sqrt{5}}{2\sqrt{2}}$. Take an arbitrary point $x = (x_1, x_2) \in L_- \setminus \{(0, 1)\}$ (points in L_+ are not to be considered) and $y = (0, 1)$. One has

$$d(x,y)=\sqrt{2}|x_1|,\quad d(y,F(x))=\sqrt{x_1^2+\left(-\frac{x_1}{2}\right)^2}=\frac{\sqrt{5}}{2}|x_1|,$$

and $d(x,F(x))=\frac{3}{2}|x_1|$, so

$$\frac{3-\sqrt{5}}{2\sqrt{2}}\sqrt{2}|x_1|+\frac{\sqrt{5}}{2}|x_1|\leqslant\frac{3}{2}|x_1|.$$

On the other hand, it results in

$$d(F(y),F(x))=d(y,F(x))=\frac{\sqrt{5}}{2}|x_1|\leqslant\frac{\sqrt{5}}{2\sqrt{2}}d(x,y).$$

Notice that the space $(\mathbb{E},d)$ is not metrically convex.

In view of further developments of the theory, it is worth mentioning that the scheme of analysis, by which the main theorem in Section 2 has been established, can be adapted in such a way to afford a directional counterpart of Theorem 1 in [16].

THEOREM 3.2. *Let $(\mathbb{E},d)$ be a complete metric space, $K\subseteq\mathbb{E}$ be closed and nonempty, and let $F\colon K\rightrightarrows\mathbb{E}$ be a set-valued map with nonempty closed values. Assume that there are a constant $L>0$, a proper (i.e. not identically $+\infty$) bounded below l.s.c. function $\psi\colon K\to(-\infty,+\infty]$, a function $q\colon(0,+\infty)\to\mathbb{R}$, and a function $p\colon[0,+\infty)\to(0,+\infty)$ with the property*

$$\inf_{s\in(0,L]}p(s)>0,$$

such that for each $x\in K$, with $x\notin F(x)$, there exists $y\in(K\cap B_L(x))\setminus\{x\}$ satisfying the inequalities

$$q(d(x,y))d(x,y)+d(y,F(x))\leqslant\psi(x)\tag{4}$$

and

$$\psi(y)+p(d(x,y))d(x,y)\leqslant\psi(x).\tag{5}$$

Then F has a fixed point.

Proof. Observe first that, without loss of generality, function ψ can be assumed to be nonnegative. Then the proof follows essentially the same lines as for Theorem 2.1. Taken an arbitrary point $x_0\in K$ and $\epsilon>f(x_0)$, it is possible to apply the Ekeland variational principle to function ψ, getting a global minimum $x_\lambda\in K$ for the $\frac{\epsilon}{\lambda}$ perturbation of ψ. Ab absurdo, assume that $\psi(x_\lambda)>0$ for every $\lambda>0$. Set $p_0=\inf_{s\in(0,L]}p(s)$ and $\lambda=\frac{2\epsilon}{p_0}$. Since corresponding to x_λ there is $y\in(K\cap B_L)\setminus\{x_\lambda\}$ fulfilling inequality (5), one obtains

$$\begin{aligned}\psi(x_\lambda)&<\psi(y)+\frac{\epsilon}{\lambda}d(x_\lambda,y)\leqslant\psi(x_\lambda)+\left[\frac{\epsilon}{\lambda}-p(d(x_\lambda,y))\right]d(x_\lambda,y)\\&\leqslant\psi(x_\lambda)-\frac{p_0}{2}d(x_\lambda,y)<\psi(x_\lambda)\end{aligned}$$

which, because of the nonnegativity assumption on ψ, forces $\psi(x_\lambda)=0$. This fact, taking into account inequality (5), gives $d(x_\lambda,y)=0$ and the last equality, along with inequality (4), entails $x_\lambda=y\in F(x_\lambda)$. Thus the proof is complete. □

Another recent generalization of the notion of metric contraction appeared in [1] under the name of weakly contractive map. The next definition can be regarded as a directional variant for the set-valued version of such notion, which was considered in [3].

DEFINITION 3.1. Let $K \subseteq \mathbb{E}$. A map $F : K \to CB(\mathbb{E})$ is called *directionally weakly contractive* if there exists $\alpha \in (0, 1], a : (0, +\infty) \to [\alpha, 1]$ and $\phi : (0, +\infty) \to (0, +\infty)$, with $\phi(t) \leqslant t$, such that for every $x \notin F(x)$, there is $y \in K \setminus \{x\}$ satisfying the inequalities

$$a(d(x, y))d(x, y) + d(y, F(x)) \leqslant d(x, F(x))$$

and

$$H_+(F(y), F(x)) \leqslant d(x, y) - \phi(d(x, y)).$$

This class of generalized multi-valued contractions can be embedded in the proposed theory inasmuch as it is a subclass of that of directional multi-valued $k(\cdot)$-contractions.

THEOREM 3.3. *Let K be a closed nonempty subset of a complete metric space $(\mathbb{E}, d)$ and let $F : K \to CB(\mathbb{E})$ be an u.s.c. directionally weakly contractive map. Suppose that*

$$\limsup_{s\to 0^+} \frac{s}{\phi(s)} < \begin{cases} +\infty, & \text{if } \exists\delta > 0 : a(s) = 1, \forall s \in (0, \delta], \\ \liminf_{s\to 0^+} \frac{1}{1-a(s)}, & \text{otherwise}, \end{cases}$$

where ϕ and a are the functions occuring in the definition of directionally weakly contractive map. If F has the almost fixed point property in K, then F admits a fixed point (in K).

Proof. Set $k(t) = 1 - t^{-1}\phi(t)$, for $t > 0$. Since $0 < \phi(t) \leqslant t$, then $k : (0, +\infty) \to [0, 1)$. Thus F is a directional multi-valued $k(\cdot)$-contraction fulfilling condition (C). Indeed, in the case in which there is $\delta > 0$ such that $a(s) = 1$ for every $s \in (0, \delta]$, or in which $\liminf_{s\to 0^+} \frac{1}{1-a(s)} = +\infty$, then by the hypothesis $\limsup_{s\to 0^+} \frac{s}{\phi(s)} < +\infty$ one has

$$\begin{aligned}\limsup_{s\to 0^+} k(s) &= 1 - \liminf_{s\to 0^+} \frac{\phi(s)}{s} = 1 - \frac{1}{\limsup_{s\to 0^+} \frac{s}{\phi(s)}} < 1 \\ &= \liminf_{s\to 0^+} a(s).\end{aligned}$$

Otherwise, if $\liminf_{s\to 0^+} \frac{1}{1-a(s)} < +\infty$, one has

$$\begin{aligned}\limsup_{s\to 0^+} k(s) &= 1 - \liminf_{s\to 0^+} \frac{\phi(s)}{s} = 1 - \frac{1}{\limsup_{s\to 0^+} \frac{s}{\phi(s)}} < 1 \\ &\quad - \frac{1}{\liminf_{s\to 0^+} \frac{1}{1-a(s)}} = \liminf_{s\to 0^+} a(s).\end{aligned}$$

So, in any case it holds $\limsup_{s\to 0^+} k(s) < \liminf_{s\to 0^+} a(s)$. Thus the thesis follows from Corollary 2.1. □

For weakly contractive maps as defined in [3], i.e. for set-valued maps $F\colon \mathbb{E} \to CB((\mathbb{E})$ for which there exists $\phi\colon [0,+\infty) \to [0,+\infty)$, with $\phi(0)=0$ and $0<\phi(t)\leqslant t$ whenever $t \in (0,+\infty)$, such that

$$H_+(F(y),F(x)) \leqslant d(x,y) - \phi(d(x,y)), \quad \forall x,y \in \mathbb{E}, \tag{6}$$

one can derive a generalization of Theorem 3.1 in [3] as an easy consequence of the Mizoguchi–Takahashi theorem.

COROLLARY 3.1. *Let* $(\mathbb{E},d)$ *be a complete metric space and let* $F\colon \mathbb{E} \to CB(\mathbb{E})$ *be an u.s.c. weakly contractive set-valued map, for which function* ϕ *occuring in the related definition is l.s.c. from the right and satisfies*

$$\limsup_{s\to 0^+} \frac{s}{\phi(s)} < +\infty.$$

Then F has a fixed point.

Proof. In a similar manner as in the proof of Theorem 3.3, one sees that F is a multi-valued $k(\cdot)$-contraction, with $k(t)=1-t^{-1}\phi(t)$. Since by assumption function $t \mapsto t^{-1}\phi(t)$ is l.s.c. from the right and ϕ takes positive values on $(0,+\infty)$, one obtains

$$\limsup_{s\to t^+} k(s) = 1 - \limsup_{s\to t^+} \frac{\phi(s)}{s} < 1 - \frac{\phi(t)}{t} < 1, \quad \forall t \in (0,+\infty).$$

Moreover, being $\limsup_{s\to 0^+} \frac{s}{\phi(s)} < +\infty$ one has

$$\limsup_{s\to t^+} k(s) < 1, \quad \forall t \in [0,+\infty).$$

It is therefore possible to apply Theorem 3.1 to get the thesis. □

REMARK 3.2. The upper semicontinuity assumption on map F not appearing in Theorem 3.1 in [3] is due to the use of H_+ instead of the Hausdorff metric $\mathcal{H}$ in (6).

In [24] multi-valued contractions were considered with a contraction factor $k(\cdot)$ expressible as a certain function of the distance from an assigned point. For such kind of multi-valued contractions a fixed point theorem extending the Nadler's one was proved relying on a proper generalization of the Ekeland variational principle, previously established. Then, by exploiting the same technique, the authors were enabled to provide a sort of "directional variant" of their main result (namely, Theorem 2.4 in [24]). In such variant map F is assumed to take compact (and therefore proximinal) values and to satisfy a directional contraction property that is typically fulfilled by multi-valued contractions acting in metrically convex spaces. Following the lines of the proposed approach, the next result is a generalization of the above mentioned directional theorem avoiding such kind of assumptions.

THEOREM 3.4. *Let K be a nonempty closed subset of a complete metric space* $(\mathbb{E},d)$. *Let* $x_0 \in \mathbb{E}$ *be an assigned point and* $\sigma \in (0,1]$ *be a constant. Let* $F\colon K \to CB(\mathbb{E})$ *be an u.s.c. map and let* $h\colon [0,+\infty) \to [0,+\infty)$ *be a continuous nondecreasing function such that*

$$\int_0^{+\infty} \frac{\mathrm{d}t}{1+h(t)} = +\infty$$

Suppose that there exists a function $a : (0, +\infty) \to (0, 1]$, *with*

$$a(t) > 1 - \frac{\sigma}{M[1+h(t)]}, \quad \forall t \in (0, +\infty)$$

for some $M > 1$, *and that for every* $x \in K$, with $x \notin F(x)$, *there exists* $y \in K \setminus \{x\}$ *such that*

$$a(d(x_0, x))d(x, y) + d(y, F(x)) \leqslant d(x, F(x))$$

and

$$H_+(F(y), F(x)) \leqslant \left(1 - \frac{\sigma}{1+h(d(x_0, x))}\right) d(x, y).$$

Then F admits a fixed point (*in K*).

Proof. Let $f : K \to \mathbb{R}_+$ be the displacement function associated to F. f is l.s.c. because F is u.s.c. Take $x_1 \in K$ and $\delta > f(x_1)$. Since (K, d) is complete and $f(x_1) < inf_{x \in K} f(x) + \delta$, it is possible to apply Lemma 1.1 [24], according to which for every $\lambda > 0$ there exists $x_\lambda \in K$ such that

$$f(x_\lambda) \leqslant f(x) + \frac{\delta}{\lambda[1 + h(d(x_0, x_\lambda))]} d(x, x_\lambda), \quad \forall x \in K. \tag{7}$$

Assume, ab absurdo, that $f(x_\lambda) > 0$ for every $\lambda > 0$, and take $\lambda = \frac{M\delta}{\sigma(M-1)}$. By hypothesis there exists $y \in K \setminus \{x_\lambda\}$ satisfying the following inequalities

$$a(d(x_0, x_\lambda))d(x_\lambda, y) + d(y, F(x_\lambda)) \leqslant f(x_\lambda)$$

and

$$H_+(F(y), F(x_\lambda)) \leqslant \left(1 - \frac{\sigma}{1+h(d(x_0, x_\lambda))}\right) d(x_\lambda, y).$$

By exploiting the two last inequalities, one obtains

$$\begin{aligned} f(y) &\leqslant d(y, F(x_\lambda)) + H_+(F(y), F(x_\lambda)) \\ &\leqslant f(x_\lambda) + \left(1 - \frac{\sigma}{1+h(d(x_0, x_\lambda))} - a(d(x_0, x_\lambda))\right) d(x_\lambda, y). \end{aligned}$$

In force of such inequality, choosing $x = y$ in (7) and replacing λ with the taken value, one finds

$$\begin{aligned} f(x_\lambda) &\leqslant f(x_\lambda) + \left(1 - \frac{\sigma}{1+h(d(x_0, x_\lambda))} - a(d(x_0, x_\lambda))\right) d(x_\lambda, y) \\ &\quad + \frac{\delta}{\frac{M\delta}{\sigma(M-1)}[1 + h(d(x_0, x_\lambda))]} d(x_\lambda, y) \\ &\leqslant f(x_\lambda) + \left\{1 - \frac{\sigma}{M[1+h(d(x_0, x_\lambda))]} - a(d(x_0, x_\lambda))\right\} d(x_\lambda, y) \\ &< f(x_\lambda) \end{aligned}$$

which leads to an absurdum. This completes the proof. □

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